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Ideal Topology on Effect Algebras

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The ideals of effect algebras induce a topology on effect algebras. The operations $\oplus$ and $\ominus$ of effect algebras are continuous with respect to this topology.

KEY WORDS: effect algebras; ideals; uniformities; topologies.

1. INTRODUCTION

Foulis and Bennet in 1994 introduced the following algebraic system $(E, \perp, \oplus, 0, 1)$ to model unsharp quantum logics, and $(E, \perp, \oplus, 0, 1)$ is said to be an *effect algebra* (Foulis and Bennet, 1994):

Let E be a set with two special elements 0, 1; $\perp$ be a subset of $E \times E$; if $(a, b) \in \perp$, denote $a \perp b$, and let $\oplus : \perp \to E$ be a binary operation; and let the following axioms hold:

(E1) *Commutative Law.* If $a, b \in E$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$.
(E2) *Associative Law.* If $a, b, c \in E$, $a \perp b$, and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(E3) *Orthocomplementation Law.* For each $a \in E$ there exists a unique $b \in E$, such that $a \perp b$ and $a \oplus b = 1$.
(E4) *Zero-Unit Law.* If $a \in E$ and $1 \perp a$, then $a = 0$.

Let $(E, \perp, \oplus, 0, 1)$ be an effect algebra. If $a, b \in E$ and $a \perp b$, we say that a and b be *orthogonal*. If $a \oplus b = 1$, we say that b is the *orthocomplement* of a, and we write $b = a^{'}$. Clearly $1^{'} = 0$, $(a^{'})^{'} = a$, $a \perp 0$, and $a \oplus 0 = a$ for all $a \in E$.

We say that $a \leq b$ if there exists $c \in E$, such that $a \perp c$ and $a \oplus c = b$. We may prove that $\leq$ is a partial ordering on E and satisfies $0 \leq a \leq 1$, $a \leq b \Leftrightarrow b^{'} \leq a^{'}$, and $a \leq b^{'} \Leftrightarrow a \perp b$ for $a, b \in E$. If the partial order $\leq$ of an effect algebra

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$(E, \perp, \oplus, 0, 1)$ defined as above is a lattice, then the effect algebra $(L, \perp, \oplus, 0, 1)$ is said to be a *lattice effect algebra*.

If $a \leq b$ and $e \in E$, such that $a \perp e$, $a \oplus e = b$, we denote e as $b \ominus a$ and $\ominus$ is called the *difference operation*. Furthermore, it is easy to prove that if $a \leq b \leq c$, then the difference operation $\ominus$ satisfies the following properties (Foulis and Bennet, 1994):

(D1) $b \ominus a \leq b$.
(D2) $b \ominus (b \ominus a) = a$.
(D3) $(c \ominus b) \leq (c \ominus a)$.
(D4) $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The properties D1–D4 are very important, in fact, Kopka and Chovanec in 1994 introduced the following quantum logic structure (Kopka and Chovanec, 1994):

A *difference poset* is a partially ordered set $(D, \leq, 0, 1)$ with a maximum element 1 and a minimum element 0, and with a partially defined binary operation $\ominus$, such that $b \ominus a$ is defined iff $a \leq b$, and the operation $\ominus$ satisfies properties (D1)–(D4).

Thus, each effect algebra can induce a difference poset. Foulis and Bennet pointed out that the converse is also true (Foulis and Bennet, 1994). Thus, the effect algebras and the difference posets are the same thing.

Let $(E, \perp, \oplus, 0, 1)$ be an effect algebra. A nonempty subset I of $(E, \perp, \oplus, 0, 1)$ is said to be an *ideal* of $(E, \perp, \oplus, 0, 1)$ if the following conditions are satisfied (Ma, Wu, and Lu):

(I1) If $x \in I$, $y \in E$, $y \leq x$ imply $y \in I$.
(I2) If $x \ominus y \in I$, $y \in I$ imply $x \in I$.

If an ideal I of effect algebra $(E, \perp, \oplus, 0, 1)$ is also a total order set, then the ideal is said to be a *total order ideal* of $(E, \perp, \oplus, 0, 1)$.

2. A TOPOLOGY OF EFFECT ALGEBRAS INDUCED BY THE IDEALS

As we knew, study of topology of effect algebras is a very important and interesting topics, see, for example, Riecanova (2000, 2002, 2004) and Qu *et al.* (2004). In 1985, Alo and Deeba showed that a base of a uniformity can be generated by the ideals of BCK-algebras (Alo and Deeba, 1985). In 1991, Qun Zhang proved the analogous result holds for BCI-algebras, and there is a lot of families of ideals, each of which determines uniformities, and so a topology of BCI-algebras (Zhang, 1991).

In this paper, we show that all the total order ideals of effect algebras can also induce a topology on effect algebras.

Let $(E, \perp, \oplus, 0, 1)$ be an effect algebra. For each ideal I of $(E, \perp, \oplus, 0, 1)$, denote $U_I = \{(x, y) : (x, y) \in E \times E$ and $x \ominus y \in I$ or $y \ominus x \in I\}$.

If $U, V \subseteq E \times E$, define $U \circ V = \{(x, z) :$ there exists $y \in X$ such that $(x, y) \in U$ and $(y, z) \in V\}$; $U^{-1} = \{(y, x) : (x, y) \in U\}$.

For each $y \in X$, denote $U(y) = \{x : (y, x) \in U\}$, and denote $\Delta = \{(x, x) | x \in X\}$.

Remark 2.1. Our definition of U_I is very different from the definition of Alo and Deeba (1985) and Zhang (1991).

A *uniformity* on effect algebra $(E, \perp, \oplus, 0, 1)$ is a nonempty family K of subsets of $E \times E$ such that the following conditions are satisfied:

(U1) $\Delta \subseteq U$ for each $U \in K$.
(U2) If $U \in K$, then $U^{-1} \in K$.
(U3) If $U \in K$, then there exits a $V \in K$, such that $V \circ V \subseteq U$.
(U4) If $U, V \in K$, then $U \cap V \in K$.
(U5) If $U \in K$ and $U \subseteq V \subseteq X \times X$, then $V \in K$.

Theorem 2.1. *Let I and J be ideals of effect algebra $(E, \perp, \oplus, 0, 1)$. Then*

(1) $\Delta = U_0$, *where* $\{0\}$ *means the zero ideal* $\{0\}$ *of* $(E, \perp, \oplus, 0, 1)$.
(2) $I \subseteq J$ *implies* $U_I \subseteq U_J$.
(3) $U_I = U_I^{-1}$.
(4) $U_I \cap U_J = U_{I \cap J}$.
(5) $U_I \cup U_J \subseteq U_I \circ U_J$.
(6) *If I is a total order ideal, then* $U_I \circ U_I = U_I$.

Proof: (1) $(x, y) \in \Delta \Leftrightarrow x \ominus y = 0$ or $y \ominus x = 0 \Leftrightarrow (x, y) \in U_0$. (2) and (3) follow from the definition immediately. (4) $(x, y) \in U_I \cap U_J \Leftrightarrow (x, y) \in U_I$ and $(x, y) \in U_J \Leftrightarrow x \ominus y \in I$ or $y \ominus x \in I$ and $x \ominus y \in J$ or $y \ominus x \in J$.

Thus, there are four possible cases:

(i) $x \ominus y \in I, x \ominus y \in J$. In this case, we get $x \ominus y \in I \cap J$.
(ii) $x \ominus y \in I, y \ominus x \in J$. Thus, we have $y \leq x$ and $x \leq y$ and so $x \ominus y = y \ominus x = 0 \in I \cap J$.
(iii) $y \ominus x \in I, x \ominus y \in J$. As (ii), we have $x = y$ and so $x \ominus y = y \ominus x = 0 \in I \cap J$.
(iv) $y \ominus x \in I, y \ominus x \in J$. In this case, we get $y \ominus x \in I \cap J$.

It follows from (i) to (iv) that $(x, y) \in U_{I \cap J}$.

(5) If $(x, y) \in U_I$, then $(x, y) \in U_I \circ U_J$, since $(y, y) \in \Delta \subseteq U_J$.
Similarly, if $(x, y) \in U_J$, then $(x, y) \in U_I \circ U_J$, too.
(6) $U_I = U_I \cup U_I \subseteq U_I \circ U_I$ from (5).

If $(x, z) \in U_I \circ U_I$, then there is $y \in X$ such that
$(x, y) \in U_I, (y, z) \in U_I$. So $x \ominus y \in I$ or $y \ominus x \in I$ and $y \ominus z \in I$ or $z \ominus y \in I$.
Thus, there are four possible cases:

(i) $x \ominus y \in I$, $y \ominus z \in I$. In this case, we have $(x \ominus z) \ominus (x \ominus y) = (y \ominus z)$, it follows from the definition of ideal that $x \ominus z \in I$.

(ii) $y \ominus x \in I$, $y \ominus z \in I$. It follows from I is a total order ideal that $y \ominus z \leq y \ominus x$ or $y \ominus x \leq y \ominus z$. If $y \ominus z \leq y \ominus x$, then $(y \ominus x) \ominus (y \ominus z) = (z \ominus x)$, we get $(z \ominus x) \leq (y \ominus x)$, so $(z \ominus x) \in I$. If $y \ominus x \leq y \ominus z$, then $(y \ominus z) \ominus (y \ominus x) = (x \ominus z)$, $(x \ominus z) \in I$.

(iii) $y \ominus x \in I, z \ominus y \in I$. In this case, we have $(z \ominus x) \ominus (y \ominus x) = (z \ominus y)$, it follows from the definition of ideal that $(z \ominus x) \in I$.

(iv) $x \ominus y \in I, z \ominus y \in I$. As (iii), we get $(x \ominus y) \ominus (z \ominus y) = (x \ominus z)$ or $(z \ominus y) \ominus (x \ominus y) = z \ominus x$, so $x \ominus z \in I$ or $z \ominus x \in I$.

Thus, we have proved that $(x, z) \in U_I$, hence $U_I \circ U_I = U_I$. This theorem is proved.

From Theorem 2.1, we can give out the first main result immediately. □

Theorem 2.2. *Let $\mathcal{I}$ be the family of all the total order ideals of effect algebra* $(E, \perp, \oplus, 0, 1)$ *and* $K_0 = \{U_I : I \in \mathcal{I}\}$. *Then*

$$K = \{V \subseteq E \times E : \exists U_I \in K_0, U_I \subseteq V\}.$$

is a uniformity of the effect algebra $(E, \perp, \oplus, 0, 1)$, *and* $T = \{U_I[x] : I \in \mathcal{I}, x \in E\}$ *is a base for the topology* τ *on* $(E, \perp, \oplus, 0, 1)$ *which is induced by the uniformity* K.

3. OPERATION CONTINUITY OF EFFECT ALGEBRAS WITH RESPECT TO TOPOLOGY τ

Riecanova (2000) studied the continuity of $\oplus$ and $\ominus$ with respect to the order topology when the effect algebras $(E, \perp, \oplus, 0, 1)$ are lattice effect algebras. Recently, Qu *et al.* (2004) studied the continuity of $\oplus$ and $\ominus$ with respect to the interval topology when the effect algebras $(E, \perp, \oplus, 0, 1)$ are also lattice effect algebras.

Now, we study the continuity of $\oplus$ and $\ominus$ with respect to the topology τ; our second main result is as follows:

Theorem 3.1. *Let* $(E, \perp, \oplus, 0, 1)$ *be an effect algebra. If the net* $\{a_\alpha\}_{\alpha \in \Lambda}$ *of* $(E, \perp, \oplus, 0, 1)$ *is τ convergent to a point a of* $(E, \perp, \oplus, 0, 1)$, *and* $a_\alpha \leq b'$ *for all* $\alpha \in \Lambda$ *and* $a \leq b'$, *then* $\{a_\alpha \oplus b\}_{\alpha \in \Lambda}$ *is τ convergent to* $a \oplus b$.

Proof: Let I be a total order ideal of $(E, \perp, \oplus, 0, 1)$. Since the net $\{a_\alpha\}_{\alpha \in \Lambda}$ is τ convergent to a, so there exists a α_0 such that for each $\alpha \geq \alpha_0$, $a_\alpha \ominus a \in I$ or $a \ominus$

$a_\alpha \in I$. Note that $a_\alpha \leq b^{'}$ for all $\alpha \in \Lambda$, $a \leq b^{'}$, so we have $a_\alpha \oplus b \ominus (a \oplus b) = a_\alpha \ominus a \in I$ or $a \oplus b \ominus (a_\alpha \oplus b) = a \ominus a_\alpha \in I$. This showed that $\{a_\alpha \oplus b\}_{\alpha \in \Lambda}$ is τ convergent to $a \oplus b$. □

Similarly, we may prove the following two theorems:

Theorem 3.2. *Let* $(E, \perp, \oplus, 0, 1)$ *be an effect algebra. If the net* $\{a_\alpha\}_{\alpha \in \Lambda}$ *of* $(E, \perp, \oplus, 0, 1)$ *is* τ *convergent to a point* a *of* $(E, \perp, \oplus, 0, 1)$*, and* $a_\alpha \leq b$ *for all* $\alpha \in \Lambda$ *and* $a \leq b$*, then* $\{b \ominus a_\alpha\}_{\alpha \in \Lambda}$ *is* τ *convergent to* $b \ominus a$.

Theorem 3.3. *Let* $(E, \perp, \oplus, 0, 1)$ *be an effect algebra. If the net* $\{a_\alpha\}_{\alpha \in \Lambda}$ *of* $(E, \perp, \oplus, 0, 1)$ *is* τ *convergent to a point* a *of* $(E, \perp, \oplus, 0, 1)$*, and* $b \leq a_\alpha$ *for all* $\alpha \in \Lambda$ *and* $b \leq a$*, then* $\{a_\alpha \ominus b\}_{\alpha \in \Lambda}$ *is* τ *convergent to* $a \ominus b$.

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